longitudinal and circumferential directions, solutions are obtained for a quarter shell. The same dimensions are used for a smooth shell; the variations of M_ϕ and M_x are given in Fig. 3, where the dashed lines represent the corresponding variations for a smooth shell.

Displacement and stress resultants under the load are given in Table 1 for $m = 41$. The results taken from Ref. 6 are given in parentheses. The radial displacements of the ribbed shell are plotted in Fig. 4.

Conclusions

To obtain sufficiently accurate figures for deflections, only a small number of significant terms is required. But the number of significant terms required to ensure a similar degree of accuracy for bending moments is much larger. It is sufficient to retain 11×11 terms in the series for deflection and 41×41 terms in the series for bending moments.

When the double trigonometric series is used to solve problems of the stress state of a smooth shell under the influence of a localized load, the number of significant terms sufficient to ensure the required degree of accuracy is much larger than the number necessary to solve an analogous problem for a ribbed shell with loads applied to the ribs. The presence of rib loads contributes to a substantial reduction and redistribution of strain and stress in the shell. The intensity of this effect increases with increasing stiffness of the ribs.

The results are in good agreement with the corresponding example in Khitrov,[6] shown in Fig. 3 and in Table 1.

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Extraction of Free-Free Modes Using Constrained Test Data

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Nomenclature

$K_{xx}, K_{xy}, K_{yx}, K_{yy}$ = partitions of stiffness matrix corresponding to the fixed and unfixed degrees of freedom
M_{axx} = analytical mass matrix of constrained structure
$M_{xx}, M_{xy}, M_{yx}, M_{yy}$ = partitions of stiffness matrix corresponding to the fixed and unfixed degrees of freedom
m = number of measured modes used in the procedure
p_e = measured modal matrix of the constrained structure
p_{al}, p_{ah} = partitions of analytical modal matrix corresponding to lower and higher frequencies
p_{el}, p_{eh} = partitions of p_e corresponding to lower and higher frequencies
Ω_{ah} = diagonal matrix of analytical higher natural frequencies of the constrained structure
Ω_e = diagonal matrix of measured natural frequencies of the constrained structure
Ω_{el}, Ω_{eh} = partitions of Ω_e corresponding to lower and higher frequencies
ω = natural frequency of the unconstrained structure
$\|(\,)\|$ = sum of the squares of all elements of matrix ( )

Introduction

It is difficult to measure the mode shapes and frequencies of a vehicle that is unconstrained (free-free) by virtue of soft supports or suspensions. Przemieniecki[1] derived an analytical method for determining the modes of an unconstrained structure using the test data from the constrained ground vibration experiment. The method requires that all of the structural modes be measured, which is not possible in most cases for large aerospace structures. Recently, Chen et al.[2] proposed an approach to improve Przemieniecki's method by modal truncation so that only the lower modes need to be measured. However, the approach does not consider any compensation for the truncation of the higher modes.

In this Note, we present a method that considers the compensation for the modal truncation effect; it is essentially the application of the approach originally developed by Baruch and Bar Itzhack,[3] Wei,[4] Baruch,[5] and Berman and Nagy[6] for stiffness matrix modification.

Method of Przemieniecki and Chen et al.

The dynamic equation for a freely vibrating unconstrained system is given by

$$\left\{\begin{bmatrix} K_{xx} & K_{xy} \\ K_{yx} & K_{yy} \end{bmatrix} - \omega^2 \begin{bmatrix} M_{xx} & M_{xy} \\ M_{yx} & M_{yy} \end{bmatrix}\right\} \begin{Bmatrix} q_x \\ q_y \end{Bmatrix} = 0 \qquad (1)$$

in which the stiffness matrices K_{xx}, $K_{xy} = K^t_{yx}$, and K_{yy} are assumed to be obtained from static tests; the mass submatrices $M_{xy} = M^t_{yx}$ and M_{yy} are analytically derived; and the mode shapes are partitioned into q_x and q_y, respectively, corresponding to the unfixed and fixed degrees of freedom during the ground constrained test. Assuming that all modes of the constrained structure are measured from tests, Przemieniecki derived the relationship between the measured frequencies Ω_e and mode shapes p_e from the constrained test and the unknown mass submatrix M_{xx}

$$M_{xx} = K_{xx} p_e \Omega_e^{-2} p_e^{-1} \qquad (2)$$

The term M_{xx} given by Eq. (2) can be substituted into Eq. (1) for the evaluation of the frequencies ω and mode shapes $\{q^t_x, q^t_y\}$ for the unconstrained system.

In Eq. (2), all of the frequencies and mode shapes are assumed to be measured from tests. To bypass this requirement, Chen et al.[2] partitioned Ω_e and p_e into lower and higher frequencies Ω_{el} and Ω_{eh} and modes p_{el} and p_{eh}, respectively, expressed M_{xx} by

$$M_{xx} = K_{xx} p_{el} \Omega_{el}^{-4} p^t_{el} K_{xx} + K_{xx} p_{eh} \Omega_{eh}^{-4} p^t_{eh} K_{xx} \qquad (3)$$

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Table 1 Comparison of errors of the extracted frequencies[a]

	Results							
				Proposed method				
Original[b]	−2.46	−2.85	−3.08	−4.86	−5.08	−6.85	−6.55	−11.99
$m = 4$	−1.02	−0.65	−0.41	−5.66	—	—	—	—
$m = 6$	−0.64	−0.44	−0.29	−0.19	0.12	−9.65	—	—
$m = 8$	−0.27	−0.19	−0.12	−0.08	−0.05	−0.03	−0.03	−16.81
				Chen et al.'s method				
$m = 4$	7.37	4.94	3.01	857.44	—	—	—	—
$m = 6$	3.10	2.22	1.51	0.95	0.55	456.12	—	—
$m = 8$	1.34	1.00	0.73	0.50	0.33	0.20	0.11	281.78

[a]Defined by $100 \times (\hat{\omega}_i - \omega_i)/\omega_i$, where $\hat{\omega}_i$ is the ith extracted natural frequency.
[b]Based on the analytical mass matrix M_{axx}.

Table 2 Comparison of errors of the extracted modes[a]

	Results							
				Proposed method				
Original	0.0010	0.0021	0.0069	0.0157	0.0269	0.0583	0.0879	0.1214
$m = 4$	0.0003	0.0003	0.0004	0.0098	—	—	—	—
$m = 6$	0.0001	0.0001	0.0001	0.0002	0.0005	0.0853	—	—
$m = 8$	0.0000	0.0000	0.0000	0.0000	0.0001	0.0002	0.0023	0.2072
				Chen et al.'s method				
$m = 4$	0.0135	0.0184	0.0265	69.7546	—	—	—	—
$m = 6$	0.0023	0.0029	0.0034	0.0041	0.0051	33.9129	—	—
$m = 8$	0.0004	0.0005	0.0006	0.0007	0.0008	0.0009	0.0011	21.6371

[a]Defined by $\|\hat{p}_{ei} - p_{ei}\|/\|p_{ei}\|$, where $\hat{p}_{ei}$ is the ith extracted mode shapes.

and, assuming $\Omega_{eh}^{-4} \ll \Omega_{el}^{-4}$, obtained an approximate M_{xx} by modal truncation,

$$M_{xx} \approx K_{xx} p_{el} \Omega_{el}^{-4} p_{el}^t K_{xx} \tag{4}$$

where the superscript t stands for matrix transpose. The numerical example presented in Ref. 2 shows that, with Eq. (4), Eq. (1) can be used to determine the first J elastic modes of the unconstrained structure accurately when $J+2$ modes and frequencies are measured. However, the frequencies of the example shown are very well separated; e.g., the ratios of the second and fifth frequencies to the first one are 2.76 and 13.49, respectively, and thus the effect of modal truncation is not significant. Such separation between frequencies is not usually observed in large space structures.

Compensation of the Modal Truncation

Equation (4) is an incomplete estimate for the mass submatrix M_{xx}, since no compensation for the truncation of the higher modes is presented. Herein we describe a method that gives an optimal estimate of M_{xx}, and as a result, the approximate analytical higher modes and frequencies will be introduced automatically to supplement the lack of the measured ones.

Assume that p_e is normalized by

$$p_e^t K_{xx} p_e = I \tag{5}$$

where I represents the unit matrix, then

$$p_e^t M_{xx} p_e = \Omega_e^{-2} \tag{6}$$

and

$$\begin{aligned} M_{xx} &= K_{xx} p_e p_e^t M_{xx} p_e p_e^t K_{xx} \\ &= K_{xx} p_{el} p_{el}^t M_{xx} p_{el} p_{el}^t K_{xx} + K_{xx} p_{eh} p_{eh}^t M_{xx} p_{eh} p_{eh}^t K_{xx} \end{aligned} \tag{7}$$

From Eq. (5), the product of the unavailable higher modes $p_{eh} p_{eh}^t$ is determined:

$$p_{eh} p_{eh}^t = K_{xx}^{-1} - p_{el} p_{el}^t \tag{8}$$

The substitution of Eqs. (6) and (8) into (7) yields

$$\begin{aligned} M_{xx} \approx{} & K_{xx} p_{el} \Omega_{el}^{-t} p_{el}^{-2} K_{xx} + \left(I - K_{xx} p_{el} p_{el}^t\right) \\ & \times M_{axx} \left(I - p_{el} p_{el}^t K_{xx}\right) \end{aligned} \tag{9}$$

in which M_{xx} in the right-hand side is replaced by the analytically evaluated mass matrix M_{axx}. The comparison of Eq. (9) with Eq. (4) indicates that the second term in Eq. (9) provides the required compensation for the truncation of the higher modes. It should be pointed out that Eq. (9) is a variation of the method originally derived for stiffness matrix modification.[3–6] That is to say, the mass matrix given by Eq. (9) is actually an optimal estimate of M_{xx} obtained from the minimization of the error function

$$\left\| K_{xx}^{-\frac{1}{2}} (M_{xx} - M_{axx}) K_{xx}^{-\frac{1}{2}} \right\| \tag{10}$$

subjected to $K_{xx} p_{el} = M_{xx} p_{el} \Omega_{el}^2$, and $M_{xx}^t = M_{xx}$.

It will be shown that, if p_{al}, the lower analytical modes of the system defined by M_{axx} and K_{xx}, expand the same modal space as p_{el}, i.e.,

$$p_{al} C = p_{el} \tag{11}$$

where C is a square transformation matrix, and $|C| \neq 0$, then Eq. (9) yields the unique solution of

$$K_{xx} [p_{el} \quad p_{ah}] = M_{xx} [p_{el} \quad p_{ah}] \begin{bmatrix} \Omega_{el}^2 & 0 \\ 0 & \Omega_{ah}^2 \end{bmatrix} \tag{12}$$

where Ω_{ah}^2 and p_{ah} are the higher analytical frequencies and modes of the system defined by M_{axx} and K_{xx}. Let $[p_{el} \quad p_{ah}]$ be normalized by

$$[p_{el} \quad p_{ah}]^t K_{xx} [p_{el} \quad p_{ah}] = \begin{bmatrix} p_{el}^t K_{xx} p_{el} & C^t p_{al}^t K_{xx} p_{ah} \\ p_{ah}^t K_{xx} p_{al} C & p_{ah}^t K_{xx} p_{ah} \end{bmatrix} = I \tag{13}$$

From Eqs. (12) and (13), we have

$$M_{xx} = K_{xx}[p_{el} \quad p_{ah}]\begin{bmatrix}\Omega_{el}^2 & 0 \\ 0 & \Omega_{ah}^2\end{bmatrix}[p_{el} \quad p_{ah}]^{-1}$$

$$= K_{xx}[p_{el} \quad p_{ah}]\begin{bmatrix}\bar{\Omega}_{el}^{-2} & 0 \\ 0 & \Omega_{ah}^{-2}\end{bmatrix}[p_{el} \quad p_{ah}]^t K_{xx}$$

$$= K_{xx} p_{el} \Omega_{el}^{-2} p_{el}^t K_{xx} + K_{xx} p_{ah} \Omega_{ah}^{-2} p_{ah}^t K_{xx}$$

$$= K_{xx} p_{el} \Omega_{el}^{-2} p_{el}^t K_{xx} + K_{xx} p_{ah} p_{ah}^t M_{axx} p_{ah} p_{ah}^t K_{xx} \qquad (14)$$

From Eq. (13), we still have $p_{ah} p_{ah}^t = K_{xx}^{-1} - p_{el} p_{el}^t$, and by substituting this relation into Eq. (14) we obtain Eq. (9) again. Therefore, it is concluded that Eq. (9) tends to replace the unavailable p_{eh} by p_{ah} and Ω_{eh} by Ω_{ah}. Generally, approximately 50% of the analytical modes can be considered acceptable, but it is not unusual that only 5% of the modes are measured. Thus, Eq. (9) provides a reasonable source for compensation for the lack of higher measured modes.

Numerical Examples

A uniform straight beam discretized by 12 elements is used as an example. Only transverse in-plane displacements are considered. The consistent and roughly lumped mass matrices are taken to be M_{xx} and M_{axx}, and the errors of the first eight nonrigid frequencies and mode shapes of the unconstrained structure evaluated with the analytical mass matrix M_{axx} are listed in Tables 1 and 2, respectively; m test modes of the constrained structure by fixing one end of the beam (cantilever beam) are then used to produce M_{xx} with Eq. (9). The errors of the extracted free-free modes by the proposed and Chen et al.'s methods are also shown in Tables 1 and 2 for comparison. It is clearly seen that the proposed method yields an improved result: it can be used with a smaller number of measured modes and gives more accurate modal data of the unconstrained structure for both frequencies and mode shapes.

Conclusions

Chen et al.[2] extended Przemieniecki's analytical method[1] for determining free-free modes from ground constrained test data to deal with the modal truncation problem. In this Note, a structural dynamic model modification method[3–6] is introduced for compensation of the modal truncation effect; the method essentially uses analytical modes in place of the unavailable measured higher ones. The improvement is expected to yield a better approximation of the dynamic model from which one can extract more accurate free- free modes and natural frequencies using the incomplete measured ones of the constrained structure. A simple example is presented to illustrate the performance of the proposed method.

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Effect of General Sparse Matrix Algorithm on Optimization of Space Structures

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Introduction

THE global stiffness matrix of a structural analysis problem is a sparse matrix with many zero elements. The number of zeros in this matrix increases with the size of the structure. The zero terms inside the stiffness matrix slow down the processing speed of the computations due to unnecessary operations performed on them. To overcome these problems, banded and skyline (envelope or variable band) methods of data structures were developed and are in use for storage of stiffness coefficients and their subsequent operations.[1] However, as the size of the structure increases, the bandwidth will also increase, and the number of zeros within the band becomes large for large structures with hundreds or thousands of members. In this case, the solution by the banded or the skyline method may not be the most efficient one. In this work, we investigate the effect of a general sparse matrix approach on the optimization of large structures using the optimality criteria approach and the Cholesky lower-upper (LU) decomposition method for the solution of the resulting linear simultaneous equations.

General Sparse Matrix Solution

In a general sparse matrix solution approach, the sparse matrix is stored in compact form with the help of some additional information about the locations of the nonzero elements in the sparse matrix.[2] By means of indirect addressing and gather operation (described later), only the nonzero elements are stored in compact form, and the arithmetic operations on zero terms are avoided. The results are then restored back to their respective positions in the original matrix by the scatter operation.

The gather operation means collecting nonzero terms of a sparse matrix and storing them in a one-dimensional array (vector) with the help of pointers to their locations in the sparse matrix. For example, in the term $x[k(i)]$, $k(i)$ is a pointer to the original index i of x in the sparse matrix. The term x is addressed as $x[k(i)]$ indirectly. This is called indirect addressing. Similarly, the scatter operation means spreading back the terms from their compact vector form to their original sparse matrix locations.

Before storing the nonzero elements in compact form, it is necessary to evaluate the fill-ins to be stored along with the nonzero terms. The fill-ins are zero terms in the original sparse matrix, but their positions will be occupied by nonzero terms after the Cholesky decomposition. To record the fill-ins and the nonzero terms, we use a logical binary variable. It is assigned a value of TRUE for nonzero and fill-in terms in the sparse stiffness matrix and FALSE for zero terms.

For every column i, the pointer to the first nonzero element is defined as first_index(i) (Fig. 1). The pointer to the diagonal element of column i is defined as diagonal_index(i) (Fig. 1). Using the two pointers and the row index (Fig. 1), we convert the noncompact sparse Cholesky factorization algorithm into an indirect addressing, compact factorization algorithm. Before doing that, the two pointers, the row index, and the value of each nonzero or fill-in term need to be recorded. The data structure used for storing the pointers and the nonzero values is illustrated in Fig. 1.

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